

Math 221-Midterm 1 Practice exam 1

1. (13 points)

- (a) (6 points) Use the definition of the derivative as a limit to find the derivative of $f(x) = x + \frac{1}{x}$ at $x = 1$.

$$\begin{aligned} f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + \frac{1}{x}) - (1 + 1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x - 1 + \frac{1}{x} - 1}{x - 1} \\ &= \lim_{x \rightarrow 1} \left(1 + \frac{\frac{1}{x} - 1}{x - 1} \right) \\ &= \lim_{x \rightarrow 1} \left(1 + \frac{1 - x}{x(x - 1)} \right) \\ &= \lim_{x \rightarrow 1} \left(1 - \frac{1}{x} \right) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

- (b) (7 points) Use the definition of the derivative as a limit to find the equation of the tangent line to the curve $y = \sqrt{x+1}$ at $x = 3$.

$$\begin{aligned} \frac{dy}{dx} \Big|_{x=3} &= \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - \sqrt{3+1}}{x - 3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} = \lim_{x \rightarrow 3} \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(x - 3)(\sqrt{x+1} + 2)} = \lim_{x \rightarrow 3} \frac{x - 3}{(x - 3)(\sqrt{x+1} + 2)} \\ &= \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{4} \end{aligned}$$

$\Rightarrow$ slope of tangent at $x=3$ is $\frac{1}{4}$

When $x=3$, $y=2 \Rightarrow$ eqn of tangent is $y = \frac{1}{4}(x-3) + 2$

2. (20 points) Calculate the derivatives of the following functions by using differentiation rules.

(a) $y = (10x + 4)^{2012}$

$$y' = 2012(10x+4)^{2011} \cdot 10$$

(b) $y = x \sin x \cos x$

$$\begin{aligned} y' &= (x \sin x)' \cos x + x \sin x (-\sin x) \\ &= (\sin x + x \cos x) \cos x - x \sin^2 x \\ &= \sin x \cos x + x \cos^2 x - x \sin^2 x \end{aligned}$$

(c) $y = \frac{x^2}{\sin(\cos x)}$

$$\begin{aligned} y' &= \frac{2x \sin(\cos x) - x^2 \cos(\cos x) \cdot (-\sin x)}{(\sin(\cos x))^2} \\ &= \frac{2x \sin(\cos x) + x^2 \cos(\cos x) \sin x}{(\sin(\cos x))^2} \end{aligned}$$

(d) $y = \frac{\sin x}{1 + \cos x}$ (Please simplify your answer as much as possible)

$$y' = \frac{\cos x (1 + \cos x) - \sin x \cdot (-\sin x)}{(1 + \cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}$$

3. (20 points) Compute the following limits or show that it does not exist. If you think the limit does not exist, explain why, and specify if the limit is ∞ or $-\infty$.

(a) $\lim_{x \rightarrow -2} \frac{|x+2|}{x(x+2)}$ $|x+2| = \begin{cases} x+2 & x \geq -2 \\ -(x+2) & x < -2 \end{cases}$

$$\lim_{x \rightarrow -2^+} \frac{|x+2|}{x(x+2)} = \lim_{x \rightarrow -2^+} \frac{x+2}{x(x+2)} = \lim_{x \rightarrow -2^+} \frac{1}{x} = -\frac{1}{2}$$

$$\lim_{x \rightarrow -2^-} \frac{|x+2|}{x(x+2)} = \lim_{x \rightarrow -2^-} \frac{-(x+2)}{x(x+2)} = \lim_{x \rightarrow -2^-} \frac{-1}{x} = \frac{1}{2}$$

Since $\lim_{x \rightarrow -2^+} \neq \lim_{x \rightarrow -2^-}$, $\lim_{x \rightarrow -2} \frac{|x+2|}{x(x+2)}$ does not exist.

(b) $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{x^2}$ (Hint: Use the double-angle formula $\cos(2x) = 1 - 2\sin^2(x)$).

$$= \lim_{x \rightarrow 0} \frac{2\sin^2 x}{x^2} = \lim_{x \rightarrow 0} 2 \left(\frac{\sin x}{x} \right)^2 = 2$$

$$(c) \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x}$$

$$-1 \leq \sin 2x \leq 1$$

$$-\frac{1}{x} \leq \frac{\sin 2x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = \lim_{x \rightarrow \infty} -\frac{1}{x} = 0$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0 \text{ by Sandwich thm}$$

$$(d) \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{(\pi - 2x)^2}$$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$= \lim_{x \rightarrow \pi/2} \frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{(\pi - 2x)^2}$$

$$= \lim_{x \rightarrow \pi/2} \frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{4\left(\frac{\pi}{2} - x\right)^2} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{4\theta^2} \quad \text{let } \theta = \frac{\pi}{2} - x$$

$$= \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{4\theta^2(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{4\theta^2(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{4\theta^2(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{1}{4} \cdot \frac{\sin^2 \theta}{\theta^2} \cdot \frac{1}{1 + \cos \theta}$$

$$= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

4. (10 points) For each of the problems, circle the (unique) correct answer.

(a) Only one of the following limits exists. Which one?

☒ A. $\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right) = 0$ because $\frac{1}{x} \rightarrow 0$

B. $\lim_{x \rightarrow \infty} \sin x$

C. $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

D. $\lim_{x \rightarrow 0} \frac{1}{\sin x}$

$$(b) \lim_{x \rightarrow -1-} \frac{x}{x^2 - x - 2} \text{ is } \lim_{x \rightarrow -1-} \frac{x \rightarrow (-1)}{(x+1)(x-2)} \quad \begin{matrix} \downarrow & \downarrow \\ 0- & (-3) \end{matrix}$$

A. ∞

☒ B. $-\infty$

C. 0.

D. None of the above.

(c) Suppose $f(x)$ is an odd function and $\lim_{x \rightarrow 1+} f(x) = 2$. What can we say about $\lim_{x \rightarrow -1+} f(x)$?

A. $\lim_{x \rightarrow -1+} f(x) = 2$

B. $\lim_{x \rightarrow -1+} f(x) = -2$

☒ C. There is not enough information to tell what $\lim_{x \rightarrow -1+} f(x)$ is.

(d) If $f(x)$ and $g(x)$ are functions defined near $x = 1$ and $\lim_{x \rightarrow 1} f(x) = 0$, then $\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = 0$

A. True

☒ B. False e.g. $\lim_{x \rightarrow 1} \frac{x-1}{2(x-1)} = \frac{1}{2}$

(e) If $f'(0)$ exists, then $\lim_{x \rightarrow 0} f(x) = f(0)$. i.e. if $f(x)$ is differentiable at $x=0$, then $f(x)$

☒ A. True

B. False

is continuous at $x=0$

5. (21 points) Do the following problems about finding asymptotes.

(a) (7 points) Find all the horizontal asymptotes of $f(x) = \frac{2x^2 + 2x}{x^2 - x - 2}$. Show your work.

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 2x}{x^2 - x - 2} = \lim_{x \rightarrow \infty} \frac{2 + \frac{2}{x}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 2$$

$$\lim_{x \rightarrow -\infty} \frac{2x^2 + 2x}{x^2 - x - 2} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{2}{x}}{1 - \frac{1}{x} - \frac{2}{x^2}} = 2$$

$\Rightarrow f(x)$ has one H.A. $y=2$

(b) (7 points) Find all the vertical asymptotes of $f(x) = \frac{2x^2 + 2x}{x^2 - x - 2}$. Show your work.

$$\begin{aligned}
 & f(x) = \frac{2x(x+1)}{(x+1)(x-2)} \text{ is not continuous at } x = -1 \text{ and } x = 2 \\
 & \underline{x = -1} \quad \lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{2x}{x-2} = \frac{-2}{-3} = \frac{2}{3} \\
 & \underline{x = 2} \quad \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{2x(x+1)}{(x+1)(x-2)} = \frac{2 \cdot 2 \cdot 3}{\downarrow 3 \quad \downarrow 0} = \infty \\
 & \Rightarrow f(x) \text{ has one V.A. } \underline{x = 2}
 \end{aligned}$$

(c) (7 points) Find the slanted asymptote of $g(x) = \frac{3x^2 + x}{x + 1}$ as $x \rightarrow \infty$. Show your work.

$$\lim_{x \rightarrow \infty} \frac{g(x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{3x^2 + x}{x+1}}{x} = \lim_{x \rightarrow \infty} \frac{3x^2 + x}{x^2 + x} = \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x}}{1 + \frac{1}{x}} = 3$$

$$\begin{aligned}
 \lim_{x \rightarrow \infty} [g(x) - 3x] &= \lim_{x \rightarrow \infty} \left(\frac{3x^2 + x}{x+1} - 3x \right) = \lim_{x \rightarrow \infty} \frac{3x^2 + x - 3x^2 - 3x}{x+1} \\
 &= \lim_{x \rightarrow \infty} \frac{-2x}{x+1} = \lim_{x \rightarrow \infty} \frac{-2}{1 + \frac{1}{x}} = -2
 \end{aligned}$$

$\Rightarrow g(x)$ is asymptotic to $y = 3x - 2$ as $x \rightarrow \infty$.

6. (16 points) Consider the function $f(x)$ defined by $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ k(x-1) + 1 & \text{if } x > 1 \end{cases}$ where k is a constant and answer the following questions.

(a) For what value (or values) of k is $f(x)$ continuous at $x = 1$? Justify your answer.

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} k(x-1) + 1 = k \cdot 0 + 1 = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1$$

$$f(1) = 1$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) \quad \text{for any } k$$

$\Rightarrow f(x)$ is continuous at $x=1$ for any value of k .

(b) For what value (or values) of k is $f(x)$ differentiable at $x = 1$? Justify your answer.

$$f(x) \text{ is diff at } x=1 \Leftrightarrow f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x-1} \text{ exists.}$$

$$\text{"Right derivative at } x=1 \text{" } \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x-1} = \frac{d}{dx} [k(x-1) + 1] \Big|_{x=1} = k$$

$$\text{"Left derivative at } x=1 \text{" } \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x-1} = \frac{d}{dx} x^2 \Big|_{x=1} = 2x \Big|_{x=1} = 2.$$

$$f'(1) \text{ exists } \Leftrightarrow k = 2$$